JOURNAL OF COMPUTATIONAL PHYSICS **101**, 207–217 (1992)

The Fourier–Chebyshev Spectral Method for Solving Two-Dimensional Unsteady Vorticity Equations

BEN-YU GUO, HE-PING MA, WEI-MING CAO, AND HUI HUANG

Shanghai University of Science and Technology, Shanghai, China

Received May 22, 1990; revised June 29, 1991

We propose a mixed method for solving the two-dimensional unsteady vorticity equations by using the Fourier-spectral approximation in the periodic direction and Chebyshev-spectral approximation in the non-periodic direction. Some numerical results are given, which are compared with those of other methods. Stability of the scheme and the optimal rate of convergence are proved. © 1992 Academic Press, Inc.

1. INTRODUCTION

There is much literature concerning the numerical solution of partial differential equations describing fluid flow. For instance, Roache [1], Raviart [2], Ben-yu Guo [3], and Canuto *et al.* [4] developed difference, finite element, and spectral methods. In studying boundary layer flow past a suddenly heated vertical plate and some other problems, we have to consider unilateral periodic problems, e.g., see Murdok [5], Ingham [6], and Moin and Kim [7]. There are several ways to solve such problems numerically. Murdok [5], Ingham [6], Macaraeg [8], and Guo and Xiong [9] proposed spectral-difference methods while Canuto *et al.* [10] and Guo and Cao [11] used spectral-finite element methods. They adopted spectral approximation in periodic directions and difference or finite element approximation in non-periodic directions. Many calculations show that such mixed methods provide better numerical results than pure difference and finite element methods.

As we know, pure spectral method has the accuracy of "infinite" order. It means that if the genuine solution of a partial differential equation is infinitely differentiable, then the error of discretization in space is of order higher than any order of N^{-1}, N being the number of the basis functions in spectral approximation. But the accuracy of both the spectral-difference method and the spectral-finite element method is still limited, due to the approximations in non-periodic directions.

In this paper, we propose another kind of mixed method for solving two-dimensional unsteady vorticity equations by using Fourier-spectral approximation in the periodic direction and Chebyshev-spectral approximation in the non-periodic direction. If M and N are the numbers of the basis functions in Chebyshev and Fourier spectral approximations, respectively, then the error of discretization is of "infinite" order. Thus such a method keeps the advantage of the pure spectral method. We shall give the scheme in Section II and the theoretical results in Section III. The numerical results are presented in Section IV. In Section V, we list some lemmas. Finally we give strict proof of the error estimation in Section VI.

II. THE SCHEME

Let $I_x = (-1, 1)$, $I_y = (0, 2\pi)$, and $\Omega = I_x \times I_y$. We denote the vorticity, stream function and kinetic viscousity by $\xi(x, y, t)$, $\psi(x, y, t)$, and $\nu > 0$, respectively. The functions $f_1(x, y, t)$, $f_2(x, y, t)$, and $\xi_0(x, y)$ are given. Let $T > 0$ and consider the following two-dimensional vorticity equations:

$$\begin{aligned} &\frac{\partial \xi}{\partial t} + J(\xi, \psi) - \nu \nabla^2 \xi = f_1, && (x, y) \in \Omega,\ t \in (0, T], \\ &-\nabla^2 \psi = \xi + f_2, && (x, y) \in \Omega,\ t \in (0, T], \\ &\xi(x, y, 0) = \xi_0(x, y), && (x, y) \in \Omega, \end{aligned} \tag{2.1}$$

where

$$J(\xi, \psi) = \frac{\partial \xi}{\partial x}\frac{\partial \psi}{\partial y} - \frac{\partial \xi}{\partial y}\frac{\partial \psi}{\partial x}.$$

We assume that all functions in (2.1) have the period 2π for the variable y, but ξ and ψ satisfy non-periodic boundary conditions for $|x| = 1$. For simplicity, we suppose that for all $y \in I_y$ and $t \leqslant T$,

$$\xi(\pm 1, y, t) = \psi(\pm 1, y, t) = 0. \tag{2.2}$$

0021-9991/92 $5.00

Let M and N be positive integers. We by $\mathbb{P}_M$ denote the set of all algebraic polynomials of degree less or equal M, and then define

$$V_M(I_x) = \{v(x) \in \mathbb{P}_M / v(-1) = v(1) = 0\}.$$

Let $V_N(I_y)$ be the set of all real trigonometric polynomials with the period 2π and the degree less than or equal to N. Define

$$S_{M,N}(\Omega) = V_M(I_x) \otimes V_N(I_y).$$

Let

$$\omega(x) = (I - x^2)^{-1/2}$$

and define the space

$$L^2_\omega(\Omega) = \{v \text{ is measurable}/(v, v)_\omega < \infty\}$$

equipped with the inner product

$$(u, v)_\omega = \frac{1}{4\pi} \int_\Omega u(x, y)\, v(x, y)\, \omega(x)\, dx\, dy.$$

Let $P_{M,N}: L^2_\omega(\Omega) \to S_{M,N}(\Omega)$ be the orthogonal projection; i.e., for any $u \in L^2_\omega(\Omega)$, the projection $P_{M,N}u \in S_{M,N}(\Omega)$ satisfies

$$(u - P_{M,N}u, v)_\omega = 0, \qquad \forall v \in S_{M,N}(\Omega).$$

Let τ be the step of the variable t and define

$$R_\tau = \{t = l\tau/2 \leqslant l \leqslant [T/\tau]\}.$$

We shall use the following central difference quotient to approximate the term $(\partial\xi/\partial t)(t)$

$$\xi_{\hat{t}}(t) = (1/2\tau)(\xi(t+\tau) - \xi(t-\tau)).$$

Let η and φ be the approximations to ξ and ψ, respectively. By using the above approximations, we obtain fully discrete Fourier–Chebyshev spectral scheme for solving (2.1)–(2.2). It is to find $(\eta(t), \varphi(t)) \in S_{M,N}(\Omega) \times S_{M,N}(\Omega)$ for all $t \in R_\tau$ such that

$$\begin{aligned}
&(\eta_{\hat{t}}(t), v)_\omega + (J(\eta(t), \varphi(t)), v)_\omega \\
&\qquad + \frac{v}{2} a_\omega(\eta(t+\tau) + \eta(t-\tau), v) \\
&\quad = (f_1(t), v)_\omega, \qquad \forall v \in S_{M,N}(\Omega), \\
&a_\omega(\varphi(t), v) = (\eta(t) + f_2(t), v)_\omega, \qquad \forall v \in S_{M,N}(\Omega), \\
&\eta(\tau) = P_{M,N}\left(\xi_0 + \tau \frac{\partial \xi}{\partial t}(0)\right), \\
&\eta(0) = P_{M,N}\xi_0,
\end{aligned} \tag{2.3}$$

where

$$a_\omega(u, v) = \frac{1}{4\pi} \int_\Omega \nabla u(x, y) \cdot \nabla(\omega(x)\, v(x, y))\, dx\, dy,$$

$$\frac{\partial \xi}{\partial t}(0) = v\nabla^2 \xi_0 - J(\xi_0, \psi_0) + f_1(0).$$

III. THEORETICAL RESULTS

For error estimations, we need some notations. We first introduce some Sobolev spaces with the weight $\omega(x)$ in I_x (see [12]). For integer $s \geqslant 0$, set

$$H^s_\omega(I_x) = \left\{ v \in L^2_\omega(I_x) / \|v\|^2_{s,\omega,I_x} = \left(\sum_{k=0}^{s} \left(\frac{d^k v}{dx^k}, \frac{d^k v}{dx^k} \right)_\omega \right)^{1/2} < \infty \right\}$$

and denote by $H^s_{0,\omega}(I_x)$ the closure of $C^\infty_0(I_x)$ in $H^s_\omega(I_x)$. For real $s > 0$, we define $H^s_\omega(I_x)$ by the complex interpolation between the spaces $H^{[s]}_\omega(I_x)$ and $H^{[s+1]}_\omega(I_x)$. Similarly, $H^s_{0,\omega}(I_x)$ denotes the complex interpolation between the spaces $H^{[s]}_{0,\omega}(I_x)$ and $H^{[s+1]}_{0,\omega}(I_x)$.

Next, let B be a Banach space with the norm $\|\cdot\|$ and I an interval in $\mathbb{R}$. Define

$$\begin{aligned}
L^2(I, B) &= \{v(z): I \to B / v \text{ is strongly measurable}, \\
&\qquad \|v\|_{L^2(I,B)} < \infty\}, \\
C(I, B) &= \{v(z): I \to B / v \text{ is strongly measurable}, \\
&\qquad |||v|||_B < \infty\},
\end{aligned}$$

where

$$\|v\|_{L^2(I,B)} = \left(\int_I \|v(z)\|^2_B\, dz \right)^{1/2},$$

$$|||v|||_B = \max_{Z \in I} \|v(z)\|_B.$$

Moreover, for all integer $\mu \geqslant 0$, define

$$H^\mu(I, B) = \{v(z) \in L^2(I, B) / \|v\|_{H^\mu(I,B)} < \infty\}$$

equipped with

$$\|v\|_{H^\mu(I,B)} = \left(\sum_{k=0}^{\mu} \left\| \frac{\partial^k v}{\partial z^k} \right\|^2_B \right)^{1/2}.$$

If real number $\mu > 0$, then we define $H^\mu(I, B)$ by the complex interpolation between the spaces $H^{[\mu]}(I, B)$ and $H^{[u+1]}(I, B)$.

For simplifying the statements of the theorems, we also define the following spaces: For $r, s \geqslant 0$, $\alpha, \beta \geqslant 1$, set

$$H^{r,s}_{\omega}(\Omega) = L^2(I_y, H^r_{\omega}(I_x)) \cap H^s(I_y, L^2_{\omega}(I_x)),$$
$$X^{\alpha,\beta}_{\omega}(\Omega) = H^{2,2}_{\omega}(\Omega) \cap H^{\alpha}(I_y, H^{\beta}_{\omega}(I_x)) \cap H^{\alpha+1}(I_y, H^{\beta-1}_{\omega}(I_x))$$

with the norms

$$\|v\|_{H^{r,s}_{\omega}(\Omega)} = \{\|v\|^2_{L^2(I_y, H^r_{\omega}(I_x))} + \|v\|^2_{H^s(I_y, L^2_{\omega}(I_x))}\}^{1/2}$$
$$\|v\|_{X^{r,s}_{\omega}(\Omega)} = \{\|v\|^2_{H^{2,2}_{\omega}(\Omega)} + \|v\|^2_{H^{\alpha}(I_y, H^{\beta}_{\omega}(I_x))} + \|v\|^2_{H^{\alpha+1}(I_y, H^{\beta-1}_{\omega}(I_x))}\}^{1/2}.$$

If $r, s \geqslant 1$, then we define

$$M^{r,s}_{\omega}(\Omega) = H^{r,s}_{\omega}(\Omega) \cap H^1(I_y, H^{r-1}_{\omega}(I_x)) \cap H^{s-1}(I_y, H^1_{\omega}(I_x))$$

with the norm

$$\|v\|_{M^{r,s}_{\omega}(\Omega)} = \{\|v\|^2_{H^{r,s}_{\omega}(\Omega)} + \|v\|^2_{H^1(I_y, H^{r-1}_{\omega}(I_x))} + \|v\|^2_{H^{s-1}(I_y, H^1_{\omega}(I_x))}\}^{1/2}.$$

Now, let $C^{\infty}_p(\Omega)$ be the set of all infinitely differentiable functions defined on $I_x \times I_y$, with the period 2π in the variable y. We denote by $H^{r,s}_{p,\omega}(\Omega)$, $M^{r,s}_{p,\omega}(\Omega)$, $X^{r,s}_{p,\omega}(\Omega)$ the closures of $C^{\infty}_p(\Omega)$ in the spaces $H^{r,s}_{\omega}(\Omega)$, $M^{r,s}_{\omega}(\Omega)$, and $X^{r,s}_{\omega}(\Omega)$, respectively. Also set

$$H^{r,s}_{0,p,\omega}(\Omega) = H^{r,s}_{p,\omega}(\Omega) \cap L^2(I_y, H^1_{0,\omega}(I_x)),$$
$$M^{r,s}_{0,p,\omega}(\Omega) = M^{r,s}_{p,\omega}(\Omega) \cap L^2(I_y, H^1_{0,\omega}(I_x)).$$

If $r = s$, we denote $H^{r,s}_{\omega}(\Omega)$, $H^{r,s}_{0,p,\omega}(\Omega)$, and $\|\cdot\|_{H^{r,s}_{\omega}(\Omega)}$ by $H^r_{\omega}(\Omega)$, $H^r_{0,p,\omega}(\Omega)$, and $\|\cdot\|_{r,\omega}$ for simplicity. The corresponding semi-norm is denoted by $|\cdot|_{r,\omega}$, etc.

In addition, we denote by $L^{\infty}(I_x)$, $L^{\infty}(\Omega)$, and $W^{1,\infty}(\Omega)$ the usual Sobolev spaces of essentially bounded functions with the norms $\|\cdot\|_{\infty,I_x}$, $\|\cdot\|_{\infty}$, and $\|\cdot\|_{1,\infty}$, respectively (see [13]).

We now consider the generalized stability of scheme (2.3). Suppose that the initial values $\eta(0)$, $\eta(\tau)$ and the right terms $f_1(t), f_2(t)$ have the errors $\tilde{\eta}(0)$, $\tilde{\eta}(\tau)$, $\tilde{f}_1(t)$, and $\tilde{f}_2(t)$, respectively, which induce the errors of $\eta(t)$ and $\varphi(t)$ denoted by $\tilde{\eta}(t)$ and $\tilde{\varphi}(t)$. Then they satisfy the equations

$$\begin{aligned}
&(\tilde{\eta}_{\hat{t}}(t), v)_{\omega} + (J(\eta(t) + \tilde{\eta}(t), \tilde{\varphi}(t)), v)_{\omega} \\
&\quad + (J(\tilde{\eta}(t), \varphi(t)), v)_{\omega} \\
&\quad + \frac{v}{2} a_{\omega}(\tilde{\eta}(t+\tau) + \tilde{\eta}(t-\tau), v) \\
&= (\tilde{f}_1(t), v)_{\omega}, \qquad \forall v \in S_{M,N}(\Omega), \\
&a_{\omega}(\tilde{\varphi}(t), v) = (\tilde{\eta}(t) + \tilde{f}_2(t), v)_{\omega}, \qquad \forall v \in S_{M,N}(\Omega).
\end{aligned} \tag{3.1}$$

For describing the errors, we introduce the notations

$$E(\tilde{\eta}, t) = \|\tilde{\eta}(t)\|^2_{\omega} + \frac{v\tau}{8} \sum_{t'=\tau}^{t-\tau} \|\tilde{\eta}(t'+\tau) + \tilde{\eta}(t'-\tau)\|^2_{1,\omega},$$
$$\rho(t) = 2\|\tilde{\eta}(0)\|^2_{\omega} + 2\|\tilde{\eta}(\tau)\|^2_{\omega} + 4\tau \sum_{t'=\tau}^{t-\tau} G_1(t')$$

with

$$G_1(t) = 2\|\tilde{f}_1(t)\|^2_{\omega} + \frac{C}{v}\|\eta(t)\|^2_{1,\infty}\|\tilde{f}_2(t)\|^2_{\omega} + \frac{C(N+M)}{v}\|\tilde{f}_2(t)\|^4_{\omega},$$

where C is a positive constant which could be different in different cases. We have the following result.

THEOREM 1. *There exist positive constants M_1 and M_2 depending only on $|||\eta|||_{1,\infty}$, $|||\varphi|||_{1,\infty}$, and v, such that if for some $t_1 \in R_{\tau}$,*

$$\rho(t_1)\, e^{2M_1 t_1} \leqslant M_2(N+M)^{-1},$$

then for all $t \in R_{\tau}$, $t \leqslant t_1$, we have

$$E(\tilde{\eta}, t) \leqslant \rho(t)\, e^{2M_1 t},$$

where

$$|||\eta|||_{1,\infty} = \max_{t \in R_{\tau}} \|\eta(t)\|_{1,\infty}, \qquad \textit{etc.}$$

We next turn to the convergence of scheme (2.3). In order to obtain optimal error estimation, we introduce the projection $P^1_{M,N}: H^1_{0,p,\omega}(\Omega) \to S_{M,N}(\Omega)$, i.e., for any $u \in H^1_{0,p,\omega}(\Omega)$, we have

$$a_{\omega}(u - P^1_{M,N}u, v) = 0, \qquad \forall v \in S_{M,N}(\Omega).$$

Now put

$$\xi^*(t) = P^1_{M,N}\xi(t), \qquad \psi^*(t) = P^1_{M,N}\psi(t).$$

Then integrating by parts, we obtain from (2.3) that

$$\begin{aligned}
&(\xi^*_{\hat{t}}(t), v)_{\omega} + (J(\xi^*(t), \psi^*(t)), v)_{\omega} \\
&\quad + \frac{v}{2} a_{\omega}(\xi^*(t+\tau) + \xi^*(t-\tau), v) \\
&= (f_1(t), v)_{\omega} + (E_1(t), v)_{\omega} \\
&\quad + v a_{\omega}(E_2(t), v) + A(v), \qquad \forall v \in S_{M,N}(\Omega), \\
&a_{\omega}(\psi^*(t), v) = (\xi^*(t) + f_2(t), v)_{\omega} \\
&\quad + (E_3(t), v)_{\omega}, \qquad \forall v \in S_{M,N}(\Omega), \\
&\xi^*(\tau) = P^1_{M,N}\xi(\tau), \\
&\xi^*(0) = P^1_{M,N}\xi_0,
\end{aligned} \tag{3.2}$$

where

$$E_1(t) = \xi_t^*(t) - \frac{\partial \xi}{\partial t}(t),$$

$$E_2(t) = \tfrac{1}{2}\xi(t+\tau) + \tfrac{1}{2}\xi(t-\tau) - \xi(t),$$

$$E_3(t) = -\xi^*(t) + \xi(t),$$

$$A(v) = (J(\xi^*(t), \psi^*(t)) - J(\xi(t), \psi(t)), v)_\omega.$$

Let

$$\tilde{\xi}(t) = \eta(t) - \xi^*(t), \qquad \tilde{\psi}(t) = \varphi(t) - \psi^*(t).$$

By substracting (3.2) from (2.3), we obtain

$$\begin{aligned}(\tilde{\xi}_t(t), v)_\omega &+ (J(\xi^*(t) + \tilde{\xi}(t), \tilde{\psi}(t)), v)_\omega \\ &+ (J(\tilde{\xi}(t), \psi^*(t)), v)_\omega \\ &+ \frac{v}{2} a_\omega(\tilde{\xi}(t+\tau) + \tilde{\xi}(t-\tau), v) \\ &= -(E_1(t), v)_\omega - v a_\omega(E_2(t), v) - A(v), \quad \forall v \in S_{M,N}(\Omega), \\ a_\omega(\tilde{\psi}(t), v) &= (\tilde{\xi}(t), v)_\omega - (E_3(t), v)_\omega, \quad \forall v \in S_{M,N}(\Omega), \\ \tilde{\xi}(\tau) &= P_{M,N}\left(\xi(0) + \tau \frac{\partial \xi}{\partial t}(0)\right) - P^1_{M,N}\xi(\tau), \\ \tilde{\xi}(0) &= P_{M,N}\xi_0 - P^1_{M,N}\xi_0. \end{aligned} \tag{3.3}$$

We have the following result.

THEOREM 2. *Let (ξ, ψ) and (η, φ) be the solutions of* (2.1) *and* (2.3), *respectively. Assume that*

(i) $\xi, \psi \in C(0, T; M^{r,s}_{0,p,\omega}(\Omega) \cap W^{1,\infty}_p(\Omega) \cap X^{\alpha,\beta}_{p,\omega}(\Omega))$,

$$\frac{\partial \xi}{\partial t} \in C(0, T; M^{r,s}_{p,\omega}(\Omega)),$$

$$\frac{\partial^2 \xi}{\partial t^2} \in L^2(0, T; H^1_{p,\omega}(\Omega)),$$

$$\frac{\partial^3 \xi}{\partial t^3} \in L^2(0, T; L^2_\omega(\Omega))$$

with $r, s \geqslant 1$, $\alpha > \frac{1}{2}$ and $\beta > \frac{5}{2}$;

(ii) *For some positive constant C_1 and C_2,*

$$C_1 N \leqslant M \leqslant C_2 N, \qquad \tau = 0((M+N)^{-1/4});$$

then there exists a positive constant M_3 depending only on ξ, ψ, and v such that for suitably small τ and M, N large enough, we have

$$\|\xi(t) - \eta(t)\|_\omega + \left(\tau \sum_{t'=\tau}^{t-\tau} \|\xi(t') - \eta(t')\|^2_{1,\omega}\right)^{1/2} \leqslant M_3(\tau^2 + M^{-r} + N^{-s}),$$

$$|\psi(t) - \varphi(t)|_{1,\omega} \leqslant M_3(\tau^2 + M^{1-r} + N^{1-s}).$$

We shall prove Theorems 1 and 2 in Section VI.

IV. NUMERICAL RESULTS

This section is devoted to numerical experiment. We give two examples.

EXAMPLE 1. Let $\Omega = (0,1) \times (0,1)$, $I_h = \{x = jh/1 \leqslant j \leqslant M-1,\ Mh = 1\}$ and $I_N = \{y = j/N/0 \leqslant j \leqslant N-1\}$. Define

$$E_\infty(t) = \max_{(x,y) \in I_h \times I_N} |\xi(x, y, t) - \eta(x, y, t)|,$$

$$E_2(t) = \left(\frac{h}{N} \sum_{(x,y) \in I_h \times I_N} |\xi(x, y, t) - \eta(x, y, t)|^2\right)^{1/2},$$

where $\eta(x, y, t)$ is the approximation to $\xi(x, y, t)$. Ben-yu Guo and Yeu-shan Xiong [9] use the spectral-difference (SD) scheme to compute problem (2.1) with two kinds of flows in the domain Ω. For comparison, we run the same examples using the Fourier–Chebyshev spectral (FCS) scheme proposed here. The results are

(i) The first flow. Let

$$\xi(x, y, t) = A \exp\{B \sin(Cx + 2\pi y) + \omega t\},$$

$$\psi(x, y, t) = A \exp\{\omega t\}(Cx + \sin 2\pi y).$$

The errors of both the SD and FCS schemes are shown in Table I for $A = C = \omega = 0.1$, $B = 0.01$, and $\tau = v = 0.001$.

(ii) The second flow. Let

$$\xi(x, y, t) = A \exp\{B \sin(Cx + 2\pi y) + \omega t\},$$

$$\psi(x, y, t) = A \exp\{\omega t\} \sin Cx \sin 2\pi y.$$

The errors are shown in Table II for $A = B = C = \omega = 0.1$ and $\tau = v = 0.001$.

EXAMPLE 2. Let $I_x = (-1, 1)$ and $I_y = (0, 2\pi)$. The test functions are

$$\xi = 0.4(x^2 - 1)(x^2 - 8) \sin 2y\, e^{1/2}, \qquad -\nabla^2 \psi = \xi.$$

For describing the errors, we define

$$E(\xi(t)) = \sqrt{\frac{\sum_{x \in \bar{I}_x} \sum_{y \in \bar{I}_y} |\xi(x, y, t) - \eta(x, y, t)|^2}{\sum_{x \in \bar{I}_x} \sum_{y \in \bar{I}_y} |\xi(x, y, t)|^2}}$$

$$E(\psi(t)) = \sqrt{\frac{\sum_{x \in \bar{I}_x} \sum_{y \in \bar{I}_y} |\psi(x, y, t) - \varphi(x, y, t)|^2}{\sum_{x \in \bar{I}_x} \sum_{y \in \bar{I}_y} |\psi(x, y, t)|^2}}$$

TABLE I

Errors for SD and FCS Schemes

	SD	FCS
$t = 1$	$M = 10, N = 4$	$M = 4, N = 4$
$E_2(t)$	0.2217×10^{-3}	0.5435×10^{-5}
$E_\infty(t)$	0.6949×10^{-3}	0.6497×10^{-5}

TABLE II

Errors for SD and FCS Schemes

	SD	FCS
$t=1$	$M=10, N=4$	$M=4, N=4$
$E_2(t)$	0.1501×10^{-3}	0.5456×10^{-5}

with

$$\tilde{I}_x=\{x_j=\cos(\pi j/N_x), j=0, 1, ..., N_x\},$$
$$\tilde{I}_y=\{y_j=\pi j/N_y, j=0, 1, ..., 2N_y\}.$$

We use the Fourier–Chebyshev spectral scheme (FCS) (2.3) to solve (2.1)–(2.2). For comparison, we also consider the Fourier spectral-linear finite element scheme (FSFE), in which I_x is uniformly partitioned with the mesh size $h=2/M$. The results are shown in Tables III and IV.

It can be seen that the results of the FCS method are much better than those of the SD method or the FSFE method. Very high accuracy solutions can be obtained with the FCS method by using only a small number of modes.

V. SOME LEMMAS

We list some lemmas which will be used in next section.

LEMMA 1. *For any $u, v\in H^1_{0,p,\omega}(\Omega)$, we have*

(i) $a_\omega(u,u)\geqslant \frac{1}{4}\|u\|^2_{1,\omega}$,

(ii) $|a_\omega(u,v)|\leqslant 2\,|u|_{1,\omega}\,|v|_{1,\omega}$.

Proof. For integer j, define

$$u_j(x)=\frac{1}{2\pi}\int_{I_y} u(x,y)\,e^{-ijy}\,dy. \tag{5.1}$$

Clearly $u_j\in H^1_{0,\omega}(I_x)$, and

$$a_\omega(u,u)=\sum_{|j|=0}\left[\left(\frac{\partial}{\partial x}u_j, \frac{\partial}{\partial x}(\omega u_j)\right)_{L^2(I_x)}+j^2(u_j,u_j)_{L^2_\omega(I_x)}\right].$$

TABLE III

Errors for FCS and FSFE Schemes with $\nu=0.001$, $\tau=0.01$.

	FCS	FSFE	
$t=5$	$M=4, N=4$	$M=4, N=4$	$M=10, N=4$
$E(\xi(t))$	0.3027×10^{-4}	0.4436×10^{-2}	0.7188×10^{-2}
$E(\psi(t))$	0.1687×10^{-4}	0.1592×10^{-1}	0.1455×10^{-2}

TABLE IV

Errors for FCS and FSFE Schemes with $\nu=0.0001$, $\tau=0.01$.

	FCS	FSFE	
$t=5$	$M=4, N=4$	$M=4, N=4$	$M=10, N=4$
$E(\xi(t))$	0.9165×10^{-4}	0.4532×10^{-2}	0.1284×10^{-1}
$E(\psi(t))$	0.8448×10^{-4}	0.1582×10^{-1}	0.1580×10^{-2}

By Lemma 2 of [14], we obtain

$$\left(\frac{\partial}{\partial x}u_j, \frac{\partial}{\partial x}(\omega u_j)\right)_{L^2(I_x)}\geqslant\frac{1}{4}\|u_j\|^2_{1,\omega,I_x}$$

and thus

$$a_\omega(u,u)\geqslant\tfrac{1}{4}\sum_{|j|=0}^{\infty}(\|u_j\|^2_{1,\omega,I_x}+j^2\|u_j\|^2_{\omega,I_x})=\tfrac{1}{4}\|u\|^2_{1,\omega}.$$

Next, we have

$$a_\omega(u,v)=\sum_{|j|=0}^{\infty}\left[\left(\frac{\partial}{\partial x}u_j, \frac{\partial}{\partial x}(\omega v_j)\right)_{L^2(I_x)}+j^2(u_j,v_j)_{L^2_\omega(I_x)}\right].$$

By Lemma 3 of [14], we have

$$\left|\left(\frac{\partial}{\partial x}u_j, \frac{\partial}{\partial x}(\omega v_j)\right)\right|\leqslant 2\,|u_j|_{1,\omega,I_x}\,|v_j|_{1,\omega,I_x}$$

and so

$$\begin{aligned}|a_\omega(u,v)|&\leqslant 2\sum_{|j|=0}^{\infty}(|u_j|_{1,\omega,I_x}|v_j|_{1,\omega,I_x}\\&\quad+j^2\|u_j\|_{\omega,I_x}\|v_j\|_{\omega,I_x})\\&\leqslant 2\left(\sum_{|j|=0}^{\infty}(|u_j|^2_{1,\omega,I_x}+j^2\|u_j\|^2_{\omega,I_x})\right)^{1/2}\\&\quad\times\left(\sum_{|j|=0}^{\infty}(|v_j|^2_{1,\omega,I_x}+j^2\|v_j\|^2_{\omega,I_x})\right)^{1/2}\\&=2\,|u_j|_{1,\omega}\,|v_j|_{1,\omega}.\end{aligned}$$

LEMMA 2. *If $u\in H^{r,s}_{0,p,\omega}(\Omega)$ and $r, s\geqslant 0$, then there exists a positive constant C independent of M, N, and u, such that*

$$\|u-P_{M,N}u\|_\omega\leqslant C(M^{-r}+N^{-s})\,|u|_{H^{r,s}_\omega(\Omega)}.$$

Proof. Let u_j be the same as in (5.1) and $P_M\colon L^2_\omega(I_x)\to V_M(I_x)$ be the orthogonal projection. Then

$$P_{M,N}u=\sum_{|j|\leqslant N}(P_M u_j(x))\,e^{ijy}.$$

By Theorem 2.1 of [10],

$$\|u_j - P_M u_j\|_{\omega, I_x} \leqslant CM^{-r} |u_j|_{r,\omega,I_x}$$

and, thus,

$$\begin{aligned}\|u - P_{M,N}u\|_{\omega}^2 &= \sum_{|j|\leqslant N} \|u_j - P_M u_j\|_{\omega,I_x}^2 + \sum_{|j|>N} \|u_j\|_{\omega,I_x}^2 \\ &\leqslant CM^{-2r} \sum_{|j|\leqslant N} |u_j|_{r,\omega,I_x}^2 \\ &\quad + CN^{-2s} \sum_{|j|>N} |j|^{2s} \|u_j\|_{\omega,I_x}^2 \\ &\leqslant CM^{-2r} |u|_{L^2(I_y, H_\omega^r(I_x))}^2 \\ &\quad + CN^{-2s} |u|_{H^s(I_y, L_\omega^2(I_x))}^2 \\ &\leqslant C(M^{-r} + N^{-s})^2 |u|_{H_\omega^{r,s}(\Omega)}^2.\end{aligned}$$

Lemma 3. *If $u \in H_{0,p,\omega}^1(\Omega) \cap M_\omega^{r,s}(\Omega)$ and $r, s \geqslant 1$, then there exists a positive constant C independent of M, N, and u, such that*

$$\|u - P_{M,N}^1 u\|_{1,\omega} \leqslant C(M^{1-r} + N^{1-s}) |u|_{M_\omega^{r,s}(\Omega)}.$$

If, in addition, for positive constants C_1 and C_2 such that $C_1 N \leqslant M \leqslant C_2 N$, then we also have

$$\|u - P_{M,N}^1 u\|_{\omega} \leqslant C(M^{-r} + N^{-s}) |u|_{M_\omega^{r,s}(\Omega)}.$$

Proof. Let u_j be the same as in (5.1) and

$$u_* = \sum_{|j|\leqslant N} (P_M^1 u_j(x)) e^{ijy},$$

where $P_M^1 u_j(x)$ is given by

$$\left(\frac{\partial}{\partial x}(u_j - P_M^1 u_j), \frac{\partial}{\partial x}(\omega v)\right)_{L^2(I_x)} = 0, \quad \forall v \in V_M(I_x).$$

According to Theorem 1.6 of [15], we have

$$\begin{aligned}&\|u_j - P_M^1 u_j\|_{\mu,\omega,I_x}^2 \\ &\quad \leqslant CM^{\mu - r} \|u_j\|_{r,\omega,I_x}, \quad \mu = 0, 1, |j| = 0, 1, \ldots.\end{aligned}$$

Using of Lemma 1, we have

$$\begin{aligned}\|u - P_{M,N}^1 u\|_{1,\omega}^2 &\leqslant C \inf_{v \in S_{M,N}(\Omega)} \|u - v\|_{1,\omega}^2 \\ &\leqslant C \|u - u_*\|_{1,\omega}^2 \\ &\leqslant C \sum_{|j|\leqslant N} (\|u_j - P_M^1 u_j\|_{1,\omega,I_x}^2 \\ &\quad + j^2 \|u_j - P_M^1 u_j\|_{\omega,I_x}^2) \\ &\quad + \sum_{|j|>N} (\|u_j\|_{1,\omega,I_x}^2 + j^2 \|u_j\|_{\omega,I_x}^2) \\ &\leqslant CM^{2(1-r)} \sum_{|j|\leqslant N} |u_j|_{r,\omega,I_x}^2 \\ &\quad + CN^{2(1-s)} \sum_{|j|>N} (|j|^{2(1-s)} \|u_j\|_{1,\omega,I_x}^2 \\ &\quad + |j|^{2s} \|u_j\|_{\omega,I_x}^2) \\ &\leqslant C(M^{1-r} + N^{1-s})^2 |u|_{M_\omega^{r,s}(\Omega)}^2.\end{aligned}$$

By means of the duality and the fact that $C_1 N \leqslant M \leqslant C_2 N$, it is not difficult to show that

$$\begin{aligned}\|u - P_{M,N}^1 u\|_{\omega} &\leqslant C(M^{-1} + N^{-1})(M^{1-r} + N^{1-s}) |u|_{M_\omega^{r,s}(\Omega)} \\ &\leqslant C(M^{-r} + N^{-s}) |u|_{M_\omega^{r,s}(\Omega)}.\end{aligned}$$

Lemma 4. *If $C_1 N \leqslant M \leqslant C_2 N$ and $u \in H_{0,p,\omega}^1(\Omega) \cap X_p^{\alpha,\beta}(\Omega)$ with $\alpha > \frac{1}{2}$, $\beta > \frac{5}{2}$, then there exists a positive constant C independent of M, N, and u, such that*

$$\|P_{M,N}^1 u\|_{1,\infty} \leqslant C \|u\|_{X^{\alpha,\beta}(\Omega)}.$$

Proof. Let

$$P_{M,N}^1 u = \sum_{|j|\leqslant N} u_j^*(x) e^{ijy}.$$

Then

$$\|P_{M,N}^1 u\|_{1,\infty} \leqslant \sum_{|j|\leqslant N} [\|u_j^*\|_{1,\infty,I_x} + |j| \|u_j^*\|_{\infty,I_x}]. \tag{5.2}$$

Let u_j be the same as in (5.1) and $\Pi_M: C(I_x) \to \mathbb{P}_M$ be the Lagrange interpolation whose interpolation points are the extreme points of Chebyshev polynomials of degree M, i.e.,

$$x_l = \cos\frac{l\pi}{M}, \qquad l = 0, 1, \ldots, M.$$

Then

$$\begin{aligned}\|u_j^*\|_{1,\infty,I_x} \leqslant \|u_j^* - \Pi_M u_j\|_{1,\infty,I_x} &+ \|u_j - \Pi_M u_j\|_{1,\infty,I_x} \\ &+ \|u_j\|_{1,\infty,I_x}.\end{aligned} \tag{5.3}$$

By the inverse inequality in $\mathbb{P}_M$ (see theorem 1 of [16]), we have

$$\|u_j\|_{\infty,I_x} \leqslant CM^{1/2} \|u_j\|_{0,I_x}.$$

By error estimation of interpolation (see Theorem 3.1 of [12]),

$$\|u_j - \Pi_M u_j\|_{m,\omega,I_x} \leqslant M^{2m-\beta} \|u_j\|_{\beta,\omega,I_x}.$$

Hence

$$\begin{aligned}\|u_j^* - \Pi_M u_j\|_{1,\infty,I_x} &\leqslant CM^{1/2}\|u_j^* - \Pi_M u_j\|_{1,\omega,I_x}\\ &\leqslant CM^{1/2}(\|u_j - u_j^*\|_{1,\omega,I_x} + \|u_j - \Pi_M u_j\|_{1,\omega,I_x})\\ &\leqslant CM^{1/2}\|u_j - u_j^*\|_{1,\omega,I_x} + CM^{5/2-\beta}\|u_j\|_{\beta,\omega,I_x}\\ &\leqslant CM^{1/2}\|u_j - u_j^*\|_{1,\omega,I_x} + C\|u_j\|_{\beta,\omega,I_x}. \qquad (5.4)\end{aligned}$$

Because of $\beta > \frac{5}{2}$ and $\omega(x) \geqslant 1$,

$$H_\omega^\beta(I_x) \subset H^\beta(I_x) \subsetneq C^2(I_x).$$

Therefore by error estimation in the maximum norm of Chebyshev polynomials (see [17]), we obtain

$$\|u_j - \Pi_M u_j\|_{1,\infty,I_x} \leqslant \frac{C\ln M}{\sqrt{M}}\|u_j\|_{C^2(I_x)} \leqslant C\|u_j\|_{\beta,\omega,I_x}. \qquad (5.5)$$

On the other hand, we have from Lemma 3 that

$$\begin{aligned}\sum_{|j|\leqslant N}\|u_j^*\|_{1,\infty,I_x} &\leqslant CM^{1/2}\sum_{|j|\leqslant N}\|u_j - u_j^*\|_{1,\omega,I_x} + C\sum_{|j|\leqslant N}\|u_j\|_{\beta,\omega,I_x}\\ &\leqslant CM^{1/2}N^{1/2}\left(\sum_{|j|\leqslant N}\|u_j - u_j^*\|^2_{1,\omega,I_x}\right)^{1/2} + C\left(\sum_{|j|\leqslant N}(1+|j|^2)^{-\alpha}\right)^{1/2}\\ &\quad\times\left(\sum_{|j|\leqslant N}(1+|j|^{2\alpha})\|u_j\|^2_{\beta,\omega,I_x}\right)^{1/2}\\ &\leqslant CM^{1/2}N^{1/2}(M^{-1}+N^{-1})\|u\|_{2,\omega} + C\|u\|_{H^\alpha(I_y,H_\omega^\beta(I_x))}\\ &\leqslant C\|u\|_{X^{\alpha,\beta}(\Omega)}. \qquad (5.6)\end{aligned}$$

By substituting (5.4)–(5.6) into (5.3), we obtain the estimation for $\|u_j^*\|_{1,\infty,I_x}$. We can estimate $|j|\ \|u_j^*\|_{\infty,I_x}$ in similar way and then complete the proof by (5.2).

LEMMA 5. *If $u, v \in H^1_{0,p,\omega}(\Omega)$ and $z \in W_p^{1,\infty}(\Omega)$, then*

$$|(J(u,z),v)_\omega| \leqslant 2\|z\|_{1,\infty}\|u\|_\omega |v|_{1,\omega}.$$

Proof. By integrating by parts, it is easy to verify that

$$\begin{aligned}|(J(u,z),v)_\omega + (J(v,z),u)_\omega| &= \left|\int_\Omega uv\frac{\partial\omega}{\partial x}\frac{\partial z}{\partial y}dx\,dy\right|\\ &\leqslant\left\|\frac{\partial z}{\partial y}\right\|_\infty\left[\int_{I_y}\left(\int_{I_x}u^2\omega\,dx\right)^{1/2}\times\left(\int_{I_x}v^2\omega^5\,dx\right)^{1/2}dy\right]. \qquad (5.7)\end{aligned}$$

By Lemma 1 of [14],

$$\int_{I_x}v^2\omega^5\,dx \leqslant \int_{I_x}\left(\frac{\partial v}{\partial x}\right)^2\omega\,dx. \qquad (5.8)$$

Thus

$$\begin{aligned}|(J(u,z),v)_\omega| &\leqslant |(J(v,z),u)_\omega| + \left\|\frac{\partial z}{\partial y}\right\|_\infty\left[\int_\Omega u^2\omega\,dx\,dy\right]^{1/2}\times\left[\int_\Omega\left(\frac{\partial v}{\partial x}\right)^2\omega\,dx\,dy\right]^{1/2}\\ &\leqslant 2\|z\|_{1,\infty}\|u\|_\omega|v|_{1,\omega}.\end{aligned}$$

LEMMA 6. *There exists a positive constant C independent of M, N such that for all $u \in S_{M,N}(\Omega)$,*

$$\|u\|_\infty \leqslant CM^{1/2}\left(\|u\|_\omega + \left\|\frac{\partial u}{\partial y}\right\|_\omega\right).$$

Proof. Let u_j be the same as in (5.1). Then $u_j \in V_M(I_x)$ and by Theorem 1 of [16],

$$\begin{aligned}\|u\|_\infty &\leqslant \sum_{|j|\leqslant N}\|u_j\|_{\infty,I_x} \leqslant CM^{1/2}\sum_{|j|\leqslant N}\|u_j\|_{\omega,I_x}\\ &\leqslant CM^{1/2}\left(\sum_{|j|\leqslant N}(1+j^2)^{-1}\right)^{1/2}\times\left(\sum_{|j|\leqslant N}(1+j^2)\|u_j\|^2_\omega\right)^{1/2}\\ &\leqslant CM^{1/2}\left(\|u\|_\omega + \left\|\frac{\partial u}{\partial y}\right\|_\omega\right).\end{aligned}$$

LEMMA 7. *There exists a positive constant C independent of M, N, u, and v, such that for all $u, v \in S_{M,N}(\Omega)$,*

$$\|J(u,v)\|^2_\omega \leqslant CN|u|^2_{1,\omega}|v|_{1,\omega}\left(|v|^2_{1,\omega} + \left\|\frac{\partial^2 v}{\partial x\,\partial y}\right\|^2_\omega\right)^{1/2}.$$

Proof. We first show that if $u \in H^{1,0}_\omega(\Omega)$, $v \in S_{M,N}(\Omega)$, and $u(-1,y)=0$ or $u(1,y)=0$, then there is a positive constant C independent of M, N, u, and v, such that

$$\|uv\|^2_\omega \leqslant C\|u\|_\omega\left\|\frac{\partial u}{\partial x}\right\|_\omega\|v\|_\omega\left(\|v\|^2_\omega + \left\|\frac{\partial v}{\partial y}\right\|^2_\omega\right)^{1/2}. \qquad (5.9)$$

In fact, we have

$$\|uv\|^2_\omega \leqslant A(u)\,B(v),$$

where

$$A(u)=\frac{1}{2\pi}\int_0^{2\pi}\sup_{x\in I_x} u^2(x,y)\,dy,$$

$$B(v)=\tfrac{1}{2}\int_{-1}^{1}\sup_{y\in I_y} v^2(x,y)\,\omega(x)\,dx.$$

Let $u(-1,y)=0$, then

$$\sup_{x\in I_x} u^2(x,y)=\sup_{x\in I_x}\int_{-1}^{x} 2u(x',y)\frac{\partial}{\partial x}u(x',y)\,dx' \leqslant 2\,\|u(\cdot,y)\|_{L^2(I_x)}\left\|\frac{\partial}{\partial x}u(\cdot,y)\right\|_{L^2(I_x)}.$$

Because $\omega(x)\geqslant 1$ for $|x|<1$, then

$$A(u)\leqslant \|u\|_\omega\left\|\frac{\partial u}{\partial x}\right\|_\omega. \tag{5.10}$$

On the other hand, we have

$$\begin{aligned}B(v)&\leqslant C\int_{-1}^{1}\|v(x,\cdot)\|_{L^2(I_y)}\|v(x,\cdot)\|_{H^1(I_y)}\,\omega(x)\,dx\\ &\leqslant C\left(\int_{-1}^{1}\|v(x,\cdot)\|^2_{L^2(I_y)}\,\omega(x)\,dx\times\int_{-1}^{1}\|v(x,\cdot)\|^2_{H^1(I_y)}\,\omega(x)\,dx\right)^{1/2}\\ &\leqslant C\,\|v\|_\omega\left(\|v\|^2_\omega+\left\|\frac{\partial v}{\partial y}\right\|^2_\omega\right)^{1/2}.\end{aligned} \tag{5.11}$$

The combination of (5.10) with (5.11) leads to (5.9).

We next turn to prove the conclusion of the lemma. Clearly

$$\|J(u,v)\|^2_\omega\leqslant 2\left(\left\|\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}\right\|^2_\omega+\left\|\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right\|^2_\omega\right). \tag{5.12}$$

In addition, $\partial u/\partial x$, $\partial u/\partial y$, $\partial v/\partial x$, $\partial v/\partial y\in S_{M,N}(\Omega)$ and

$$\frac{\partial u}{\partial y}(-1,y)=\frac{\partial v}{\partial y}(-1,y)=0.$$

Hence we have from (5.9) that

$$\left\|\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}\right\|^2_\omega\leqslant C\left\|\frac{\partial v}{\partial y}\right\|_\omega\left\|\frac{\partial^2 v}{\partial x\,\partial y}\right\|_\omega\left\|\frac{\partial u}{\partial x}\right\|_\omega\times\left(\left\|\frac{\partial u}{\partial x}\right\|^2_\omega+\left\|\frac{\partial^2 u}{\partial x\,\partial y}\right\|^2_\omega\right)^{1/2},$$

$$\left\|\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right\|^2_\omega\leqslant C\left\|\frac{\partial u}{\partial y}\right\|_\omega\left\|\frac{\partial^2 u}{\partial x\,\partial y}\right\|_\omega\left\|\frac{\partial v}{\partial x}\right\|_\omega\times\left(\left\|\frac{\partial v}{\partial x}\right\|^2_\omega+\left\|\frac{\partial^2 v}{\partial x\,\partial y}\right\|^2_\omega\right)^{1/2}.$$

By substituting the above two inequalities into (5.12), we obtain

$$\begin{aligned}\|J(u,v)\|^2_\omega&\leqslant C\Bigg[\left\|\frac{\partial v}{\partial y}\right\|_\omega\left\|\frac{\partial^2 v}{\partial x\,\partial y}\right\|_\omega\left\|\frac{\partial u}{\partial x}\right\|_\omega\times\left(\left\|\frac{\partial u}{\partial x}\right\|^2_\omega+\left\|\frac{\partial^2 u}{\partial x\,\partial y}\right\|^2_\omega\right)^{1/2}\\ &\quad+\left\|\frac{\partial u}{\partial y}\right\|_\omega\left\|\frac{\partial^2 u}{\partial x\,\partial y}\right\|_\omega\left\|\frac{\partial v}{\partial x}\right\|_\omega\times\left(\left\|\frac{\partial v}{\partial x}\right\|^2_\omega+\left\|\frac{\partial^2 v}{\partial x\,\partial y}\right\|^2_\omega\right)^{1/2}\Bigg]\\ &\leqslant C\Bigg[(1+N^2)^{1/2}\left\|\frac{\partial v}{\partial y}\right\|_\omega\left\|\frac{\partial u}{\partial x}\right\|_\omega\times\left\|\frac{\partial u}{\partial x}\right\|_\omega\left\|\frac{\partial^2 v}{\partial x\,\partial y}\right\|_\omega\\ &\quad+N\left\|\frac{\partial u}{\partial y}\right\|_\omega\left\|\frac{\partial u}{\partial x}\right\|_\omega\left\|\frac{\partial v}{\partial x}\right\|_\omega\times\left(\left\|\frac{\partial v}{\partial x}\right\|^2_\omega+\left\|\frac{\partial^2 v}{\partial x\,\partial y}\right\|^2_\omega\right)^{1/2}\Bigg]\\ &\leqslant CN\,|u|^2_{1,\omega}\,|v|_{1,\omega}\left(|v|^2_{1,\omega}+\left\|\frac{\partial^2 v}{\partial x\,\partial y}\right\|^2_\omega\right)^{1/2}.\end{aligned}$$

LEMMA 8. *Let $\tilde{f}\in L^2_\omega(\Omega)$ and $u\in S_{M,N}(\Omega)$ be the solution of*

$$a_\omega(u,v)=(\tilde{f},v)_\omega,\qquad \forall v\in S_{M,N}(\Omega). \tag{5.13}$$

Then

$$\|u\|^2_{1,\omega}+\left\|\frac{\partial u}{\partial y}\right\|^2_{1,\omega}\leqslant C\,\|\tilde{f}\|^2_\omega.$$

Proof. We take $v=u$ in (5.13). By using Lemma 1 and Poincaré inequality, we obtain

$$\|u\|^2_{1,\omega}\leqslant C\,\|\tilde{f}\|^2_\omega.$$

Let

$$u=\sum_{|j|\leqslant N}u_j(x)\,e^{ijy},\qquad \tilde{f}=\sum_{|j|\leqslant N}\tilde{f}_j(x)\,e^{ijy}.$$

Then $u_j\in S_M(I_x)$ for all $|j|\leqslant N$. By putting $v=u_j(x)\,e^{ijy}$ in (5.13), we obtain

$$\left(\frac{\partial}{\partial x}u_j,\frac{\partial}{\partial x}(\omega u_j)\right)_{I_x}+j^2(u_j,\omega u_j)_{I_x}=(\tilde{f}_j,\omega u_j)_{I_x}. \tag{5.14}$$

From Lemma 8 of [14],

$$\left(\frac{\partial}{\partial x} u_j, \frac{\partial}{\partial x}(\omega u_j)\right)_{I_x} \geqslant \frac{1}{4} |u_j|^2_{1,\omega,I_x}.$$

Moreover,

$$(\tilde{f}_j, \omega u_j)_{I_x} \leqslant \frac{1}{2} j^2 \|u_j\|^2_{\omega,I_x} + \frac{1}{2j^2} \|\tilde{f}_j\|^2_{\omega,I_x}, \quad |j| \neq 0.$$

Thus (5.14) reads

$$\frac{1}{4} |u_j|^2_{1,\omega,I_x} + \frac{1}{2} j^2 \|u_j\|^2_{\omega,I_x} \leqslant \frac{1}{2j^2}, \|\tilde{f}_j\|^2_{\omega,I_x}, \quad |j| \neq 0.$$

Therefore,

$$\left\|\frac{\partial u}{\partial y}\right\|^2_{1,\omega} = \sum_{|j| \leqslant N} [j^2 |u_j|^2_{1,\omega,I_x} + j^4 \|u_j\|^2_{\omega,I_x}]$$
$$\leqslant C \sum_{|j| \leqslant N} \|\tilde{f}_j\|^2_{\omega,I_x} \leqslant C \|\tilde{f}\|^2_\omega.$$

LEMMA 9. (see Lemma 4.16 of [3]). *Suppose that the following conditions are fulfilled*:

(i) *Z is a non-negative function on R_τ*;

(ii) *D_1, D_2 and q are non-negative constants*;

(iii) *For all $t \in R_\tau$ and $t \geqslant \tau$,*

$$Z(t) \leqslant q + \tau \sum_{t' \leqslant t-\tau} [D_1 Z(t') + D_2 Z^2(t')];$$

(iv) *$Z(\tau) \leqslant q$ and $qe^{2D_1 t_1} \leqslant D_1/D_2$ for some $t_1 \in R_\tau$, then for all $t \in R_\tau$ and $t \leqslant t_1$, we have*

$$Z(t) \leqslant qe^{2D_1 t}.$$

VI. THE PROOFS OF THEOREMS

Proof of Theorem 1. We take $v = \tilde{\eta}(t+\tau) + \tilde{\eta}(t-\tau)$ in the first formula of (3.1). By Lemma 1 and the identity

$$(\tilde{\eta}_{\hat{t}}(t), \tilde{\eta}(t+\tau) + \tilde{\eta}(t-\tau))_\omega = (\|\tilde{\eta}(t)\|^2_\omega)_{\hat{t}},$$

we obtain

$$(\|\tilde{\eta}(t)\|^2_\omega)_{\hat{t}} + \frac{\nu}{8} \|\tilde{\eta}(t+\tau) + \tilde{\eta}(t-\tau)\|^2_{1,\omega} + \sum_{l=1}^{3} F_l(t)$$
$$\leqslant \frac{1}{8} \|\tilde{\eta}(t+\tau) + \tilde{\eta}(t-\tau)\|^2_\omega + 2 \|\tilde{f}_1(t)\|^2_\omega, \tag{6.1}$$

where

$$F_1(t) = |(J(\eta(t), \tilde{\varphi}(t)), \tilde{\eta}(t+\tau) + \tilde{\eta}(t-\tau))_\omega|,$$
$$F_2(t) = |(J(\tilde{\eta}(t), \varphi(t)), \tilde{\eta}(t+\tau) + \tilde{\eta}(t-\tau))_\omega|,$$
$$F_3(t) = |(J(\tilde{\eta}(t), \tilde{\varphi}(t)), \tilde{\eta}(t+\tau) + \tilde{\eta}(t-\tau))_\omega|.$$

By Lemma 8 and the second formula of (3.1),

$$|\tilde{\varphi}(t)|^2_{1,\omega} + \left\|\frac{\partial^2}{\partial x\,\partial y} \tilde{\varphi}(t)\right\|^2_\omega$$
$$\leqslant C(\|\tilde{\eta}(t)\|^2_\omega + \|\tilde{f}_2(t)\|^2_\omega). \tag{6.2}$$

Now, we are going to estimate $|F_l(t)|$. Clearly we obtain from (6.2) that

$$F_1(t) \leqslant \|\tilde{\eta}(t+\tau) + \tilde{\eta}(t-\tau)\|_\omega \|J(\eta(t), \tilde{\varphi}(t))\|_\omega$$
$$\leqslant C \|\tilde{\eta}(t+\tau) + \tilde{\eta}(t-\tau)\|_{1,\omega}$$
$$\times \|\eta(t)\|_{1,\infty} |\tilde{\varphi}(t)|_{1,\omega}$$
$$\leqslant \frac{\nu}{32} \|\tilde{\eta}(t+\tau) + \tilde{\eta}(t-\tau)\|^2_{1,\omega}$$
$$+ \frac{C}{\nu} \|\eta(t)\|^2_{1,\infty} (\|\tilde{\eta}(t)\|^2_\omega + \|\tilde{f}_2(t)\|^2_\omega). \tag{6.3}$$

We have from Lemma 5 that

$$F_2(t) \leqslant C \|\varphi(t)\|_{1,\infty} \|\tilde{\eta}(t)\|_\omega \|\tilde{\eta}(t+\tau) + \tilde{\eta}(t-\tau)\|^2_{1,\omega}$$
$$\leqslant \frac{\nu}{32} \|\tilde{\eta}(t+\tau) + \tilde{\eta}(t-\tau)\|^2_{1,\omega}$$
$$+ \frac{C}{\nu} \|\varphi(t)\|^2_{1,\infty} \|\tilde{\eta}(t)\|^2_\omega. \tag{6.4}$$

Furthermore, by (5.7), (5.8), (6.2), Lemma 6, and Lemma 7, we obtain

$$F_3(t) \leqslant |(J(\tilde{\eta}(t+\tau) + \tilde{\eta}(t-\tau), \tilde{\varphi}(t)), \tilde{\eta}(t))_\omega|$$
$$+ \left\|\frac{\partial}{\partial y} \tilde{\varphi}(t)\right\|_\infty \|\tilde{\eta}(t)\|_\omega \|\tilde{\eta}(t+\tau) + \tilde{\eta}(t-\tau)\|_{1,\omega}$$
$$\leqslant CN^{1/2} \|\tilde{\eta}(t)\|_\omega \|\tilde{\eta}(t+\tau) + \tilde{\eta}(t-\tau)\|_{1,\omega}$$
$$\times \left(|\tilde{\varphi}(t)|_{1,\omega} + \left\|\frac{\partial}{\partial y} \tilde{\varphi}(t)\right\|_{1,\omega}\right)$$
$$+ CM^{1/2} \left(\|\tilde{\varphi}(t)\|_{1,\omega} + \left\|\frac{\partial}{\partial y} \tilde{\varphi}(t)\right\|_{1,\omega}\right)$$
$$\times \|\tilde{\eta}(t)\|_\omega \|\tilde{\eta}(t+\tau) + \tilde{\eta}(t-\tau)\|_{1,\omega}$$
$$\leqslant \frac{\nu}{32} \|\tilde{\eta}(t+\tau) + \tilde{\eta}(t-\tau)\|^2_{1,\omega}$$
$$+ \frac{C(N+M)}{\nu} (\|\tilde{\eta}(t)\|^4_\omega + \|\tilde{f}_2(t)\|^4_\omega). \tag{6.5}$$

By substituting (6.3)–(6.5) into (6.1), we have

$$(\|\tilde{\eta}(t)\|_\omega^2)_{\hat{t}}+\frac{\nu}{32}\|\tilde{\eta}(t+\tau)+\tilde{\eta}(t-\tau)\|_{1,\omega}^2$$
$$\leqslant\frac{1}{8}\|\tilde{\eta}(t+\tau)+\tilde{\eta}(t-\tau)\|_\omega^2$$
$$+C_1\|\tilde{\eta}(t)\|_\omega^2+C_2\|\tilde{\eta}(t)\|_\omega^4+G_1(t), \qquad (6.6)$$

where

$$C_1=\frac{C}{\nu}(\|\|\eta\|\|_{1,\infty}^2+\|\|\varphi\|\|_{1,\infty}^2), \qquad C_2=\frac{C}{\nu}(N+M),$$

$$G_1(t)=2\|\tilde{f}_1(t)\|_\omega^2+\frac{C}{\nu}\|\eta(t)\|_{1,\infty}^2\|\tilde{f}_2(t)\|_\omega^2$$
$$+\frac{C(N+M)}{\nu}\|\tilde{f}_2(t)\|_\omega^4.$$

We sum (6.6) for all $t'\in R_\tau$, $t'\leqslant t-\tau$ to obtain

$$\|\tilde{\eta}(t)\|_\omega^2+\|\tilde{\eta}(t-\tau)\|_\omega^2+\frac{\nu\tau}{16}\sum_{t'=\tau}^{t-\tau}\|\tilde{\eta}(t+\tau)+\tilde{\eta}(t'-\tau)\|_{1,\omega}^2$$
$$\leqslant\|\tilde{\eta}(0)\|_\omega^2+\|\tilde{\eta}(\tau)\|_\omega^2$$
$$+\frac{\tau}{4}\sum_{t'=\tau}^{t-\tau}\|\tilde{\eta}(t+\tau)+\tilde{\eta}(t'-\tau)\|_\omega^2$$
$$+2\tau\sum_{t'=\tau}^{t-\tau}[C_1\|\tilde{\eta}(t')\|_\omega^2$$
$$+C_2\|\tilde{\eta}(t')\|_\omega^2+G_1(t')]. \qquad (6.7)$$

Let $\tau\leqslant 1$. Because

$$\|\tilde{\eta}(t+\tau)+\tilde{\eta}(t-\tau)\|_\omega^2\leqslant 2\|\tilde{\eta}(t+\tau)\|_\omega^2+2\|\tilde{\eta}(t-\tau)\|_\omega^2,$$

then (6.7) leads to

$$E(\tilde{\eta},t)\leqslant\rho(t)+4\tau\sum_{t'=\tau}^{t-\tau}[(C_1+1)E(\tilde{\eta},t')+C_2E^2(\tilde{\eta},t')].$$

Finally, we complete the proof of Theorem 1 by applying Lemma 9 with

$$D_1=4(C_1+1), \quad D_2=4C_2, \quad q=\rho(t).$$

Proof of Theorem 2. It is easy to obtain from Lemmas 1 and 5 that

$$|\nu a_\omega(E_2(t),v)|\leqslant\frac{\nu}{128}|v|_{1,\omega}^2+128\nu|E_2(t)|_{1,\omega}^2,$$

$$|A(v)|\leqslant|(J(\xi(t)-\xi^*(t),\psi^*(t)),v)_\omega|$$
$$+|(J(\xi(t),\psi(t)-\psi^*(t)),v)_\omega|$$
$$=|(J(\xi(t)-\xi^*(t),\psi^*(t)),v)_\omega|$$
$$+|(J(\psi(t)-\psi^*(t),\xi(t)),v)_\omega|$$
$$\leqslant|v|_{1,\omega}(\|\psi^*(t)\|_{1,\infty}\|\xi(t)-\xi^*(t)\|_\omega$$
$$+\|\xi(t)\|_{1,\infty}\|\psi(t)-\psi^*(t)\|_\omega)$$
$$\leqslant\frac{\nu}{128}|v|_{1,\omega}^2+\frac{C}{\nu}(\|\psi^*(t)\|_{1,\infty}^2$$
$$\times\|\xi(t)-\xi^*(t)\|_\omega^2$$
$$+\|\xi(t)\|_{1,\infty}^2\|\psi(t)-\psi^*(t)\|_\omega^2).$$

We take $v=\tilde{\xi}(t+\tau)+\tilde{\xi}(t-\tau)$ in (3.3). By an argument similar to that in the derivation of (6.6), we obtain

$$(\|\tilde{\xi}(t)\|_\omega^2)_{\hat{t}}+\frac{\nu}{64}\|\tilde{\xi}(t+\tau)+\tilde{\xi}(t-\tau)\|_{1,\omega}^2$$
$$\leqslant\frac{1}{8}\|\tilde{\xi}(t+\tau)+\tilde{\xi}(t-\tau)\|_\omega^2+C_1^*\|\tilde{\xi}(t)\|_\omega^2$$
$$+C_2^*\|\tilde{\xi}(t)\|_\omega^4+G_2(t), \qquad (6.8)$$

where

$$C_1^*=\frac{C}{\nu}(\|\|\xi^*\|\|_{1,\infty}^2+\|\|\psi^*\|\|_{1,\infty}^2), \quad C_2^*=C_2,$$

$$G_2(t)=\|E_1(t)\|_\omega^2+128\nu|E_2(t)|_{1,\omega}^2$$
$$+\frac{C}{\nu}\|\xi^*(t)\|_{1,\infty}^2\|E_3(t)\|_\omega^2+\frac{C}{\nu}(N+M)\|E_3(t)\|_\omega^4$$
$$+\frac{C}{\nu}(\|\psi^*(t)\|_{1,\infty}^2\|\xi(t)-\xi^*(t)\|_\omega^2$$
$$+\|\xi(t)\|_{1,\infty}^2\|\psi(t)-\psi^*(t)\|_\omega^2).$$

So far, we can obtain a conclusion similar to Theorem 1. In order to complete the proof, we only have to estimate C_1^*, $\|\tilde{\xi}(0)\|_\omega$, $\|\tilde{\xi}(\tau)\|_\omega$, and $G_2(t)$. First, Lemma 4 leads to

$$\|\|\xi^*\|\|_{1,\infty}\leqslant C\|\|\xi\|\|_{X^{\alpha,\beta}(\Omega)},$$
$$\|\|\psi^*\|\|_{1,\infty}\leqslant C\|\|\psi\|\|_{X^{\alpha,\beta}(\Omega)}.$$

Second, Lemma 2 and Lemma 3 lead to

$$\|\tilde{\xi}(0)\|_\omega\leqslant\|\xi(0)-P_{M,N}\xi(0)\|_\omega+\|\xi(0)-P_{M,N}^1\xi(0)\|_\omega$$
$$\leqslant C(M^{-r}+N^{-s})\|\xi(0)\|_{M_\omega^{r,s}(\Omega)}.$$

Similarly, we have from Taylor formula that

$$\begin{aligned}\|\tilde{\xi}(\tau)\|_\omega &\leqslant \|\xi(\tau)-P^1_{M,N}\xi(\tau)\|_\omega \\ &\quad+\left\|\xi(\tau)-\xi(0)-\tau\frac{\partial}{\partial t}(0)\right\|_\omega \\ &\quad+\|\xi(0)-P_{M,N}\xi(0)\|_\omega \\ &\quad+\tau\left\|\frac{\partial\xi}{\partial t}(0)-P_{M,N}\left(\frac{\partial\xi}{\partial t}(0)\right)\right\|_\omega \\ &\leqslant C(M^{-r}+N^{-s})\left(\|\xi(\tau)\|_{M^{r,s}_\omega(\Omega)}\right. \\ &\quad\left.+\|\xi(0)\|_{H^{r,s}_\omega(\Omega)}+\left\|\frac{\partial\xi}{\partial t}(0)\right\|_{H^{r,s}_\omega(\Omega)}\right) \\ &\quad+C\tau^2\left\|\frac{\partial^2\xi}{\partial t^2}\right\|_{C(0,T;L^2_\omega(\Omega))},\end{aligned}$$

$$\begin{aligned}\tau\sum_{t'=\tau}^{t-\tau}\|E_1(t')\|^2_\omega &\leqslant 2\tau\sum_{t'=\tau}^{t-\tau}\left(\|\xi_{\hat{t}}(t')-\xi^*_{\hat{t}}(t')\|^2_\omega\right. \\ &\quad\left.+\left\|\xi_{\hat{t}}(t')-\frac{\partial\xi}{\partial t}(t')\right\|^2_\omega\right) \\ &\leqslant C\tau(M^{-r}+N^{-s})^2\sum_{t'=\tau}^{t-\tau}\|\xi_{\hat{t}}(t')\|^2_{M^{r,s}_\omega(\Omega)} \\ &\quad+C\tau^4\sum_{t'=\tau}^{t-\tau}\int_{t'-\tau}^{t'+\tau}\left\|\frac{\partial^3\xi}{\partial t^3}(t'')\right\|^2_\omega dt'' \\ &\leqslant C(M^{-r}+N^{-s})^2\left\|\frac{\partial\xi}{\partial t}\right\|^2_{L^2(0,T;M^{r,s}_\omega(\Omega))} \\ &\quad+C\tau^4\left\|\frac{\partial^3\xi}{\partial t^3}\right\|_{L^2(0,T;L^2_\omega(\Omega))},\end{aligned}$$

$$\begin{aligned}\tau\sum_{t'=\tau}^{t-\tau}|E_2(t')|^2_{1,\omega} &\leqslant C\tau^4\sum_{t'=\tau}^{t-\tau}\int_{t'-\tau}^{t'+\tau}\left|\frac{\partial^2\xi}{\partial t^2}(t'')\right|^2_{1,\omega}dt'' \\ &\leqslant C\tau^4\left\|\frac{\partial^2\xi}{\partial t^2}\right\|^2_{L^2(0,T;H^1_\omega(\Omega))},\end{aligned}$$

$$\begin{aligned}\|E_3(t)\|^2_\omega &\leqslant C(M^{-r}+N^{-s})^2\,\|\xi(t)\|^2_{M^{r,s}_\omega(\Omega)} \\ &\leqslant C(M^{-r}+N^{-s})\,|||\xi|||^2_{M^{r,s}_\omega(\Omega)}\end{aligned}$$

and

$$\begin{aligned}&\|\xi(t)-\xi^*(t)\|^2_\omega+\|\psi(t)-\psi^*(t)\|^2_\omega \\ &\quad\leqslant C(M^{-r}+N^{-s})^2\left(|||\xi|||^2_{M^{r,s}_\omega(\Omega)}+|||\psi|||^2_{M^{r,s}_\omega(\Omega)}\right).\end{aligned}$$

Thus

$$\begin{aligned}&\|\tilde{\xi}(0)\|^2_\omega+\|\tilde{\xi}(\tau)\|^2_\omega+\tau\sum_{t'=0}^{t-\tau}G_2(t') \\ &\quad\leqslant M_4(\tau^4+M^{-2r}+N^{-2s}),\end{aligned}$$

where M_4 is a positive constant depending only on $|||\xi|||_{1,\infty}$, $|||\xi|||_{M^{r,s}_\omega(\Omega)}$, $|||\xi|||_{X^{\alpha,\beta}_\omega(\Omega)}$, $|||\partial\xi/\partial t|||_{M^{r,s}_\omega(\Omega)}$, $\|\partial^2\xi/\partial t^2\|_{L^2(0,T;H^1_\omega(\Omega))}$, $\|\partial^3\xi/\partial t^3\|_{L^2(0,T;L^2_\omega(\Omega))}$, $|||\psi|||_{1,\infty}$, $|||\psi|||_{M^{r,s}_\omega(\Omega)}$, $|||\psi|||_{X^{\alpha,\beta}_\omega(\Omega)}$, and ν.

On the other hand, we use the triangle inequality to obtain

$$\|\xi(t)-\eta(t)\|_{l,\omega}\leqslant\|\xi(t)-\xi^*(t)\|_{l,\omega}+\|\tilde{\xi}(t)\|_{l,\omega},\qquad l=0,1,$$

$$|\psi(t)-\varphi(t)|_{1,\omega}\leqslant|\psi(t)-\psi^*(t)|_{1,\omega}+|\tilde{\psi}(t)|_{1,\omega}.$$

By putting the above estimations together, we complete the proof.

VII. DISCUSSION

It is shown in [9] that the spectral-difference method is better than the full difference method. But the accuracy is still limited by the order of the difference approximation. In this paper, we use the Chebyshev-spectral method in the direction of non-periodicity. Thus the method keeps the advantage of "infinite" order and solve the same problem with a tremendous gain in accuracy as shown both by the theoretical results and numerical results.

In order to save compution, the Fourier–Chebyshev pseudospectral method should be used, in which the non-linear convective term is treated by the collocation method. Using this method, we have also run the examples in Section IV and found that the accuracy of the pseudo-spectral method is nearly the same as the spectral method.

REFERENCES

1. P. J. Roache, *Computational fluid dynamics* (Hermosa, Albuquerque, NM, 1976).
2. P. A. Raviart, "Approximation numérique des phénomènes de diffusion convection," Méthod d'éléments finis en méchanics des fluids, Cours a l'école d'été d'analyse numérique (1979).
3. B. Y. Guo, *Finite Differences Methods for Partial Differential Equations* (Science Press, Beijing, 1988).
4. C. Canuto, M. Y. Hussaini, A. Quarteroni, and T. A. Zang, *Spectral Method in Fluid Dynamics* (Springer-Verlag, Berlin, 1988).
5. J. W. Murdok, AIAA Paper 86-0434, 1986 (unpublished).
6. D. B. Ingham, *Proc. R. Soc. London A* **402**, 109 (1985).
7. P. Moin and J. Kim, *J. Fluid Mech.* **118**, 341 (1982).
8. M. G. Macaraeg, *J. Comput. Phys.* **62**, 297 (1986).
9. B. Y. Guo and Y. S. Xiong, *J. Comput. Phys.* **84**, 259 (1989).
10. C. Canuto, Y. Maday, and A. Quarteroni, *Numer. Math.* **44**, 201 (1984).
11. B. Y. Guo and W. M. Cao, *Acta Math. Appl. Sin.* **7**, 257 (1991).
12. C. Canuto and A. Quarteroni, *Math. Comput.* **38**, 67 (1982).
13. R. A. Adams, *Sobolev spaces* (Academic Press, New York, 1975).
14. H. P. Ma and B. Y. Guo, *J. Comput. Math.* **6**, 48 (1988).
15. Y. Maday and A. Quarteroni, *Numer. Math.* **37**, 321 (1981).
16. A. Quarteroni, *Jpn. J. Appl. Math.* **1**, 173 (1984).
17. И. П. Натансон, *Конструкти ная теория фуникций* (ТИТТЛ, Ленинград, 1949).